

Investigation of Rheological Parameters on the Vertical Rise of a Pair of Bubbles in Viscoplastic Fluid Using Vibration

Saeed Tavangar Roosta*, Seyed Alireza Pavarpoor, Hosein Sour, Seyed Hadi Motamedshariati, Shahram Ghanbari Pakdehi

Faculty of Chemistry and Chemical Engineering, Malek Ashtar University of Technology, Tehran, Iran

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Abstract This study presents a numerical investigation into the vertical rise dynamics of a vertically aligned pair of gas bubbles rising through a viscoplastic fluid under vertical sinusoidal vibration. The two-phase hydrodynamic behavior is resolved using the Volume of Fluid (VOF) method on a high-resolution uniform structured mesh in *OpenFOAM*. The computational domain (100×100 mm, width×height) is filled with liquid up to a 70% height and contains two vertically aligned 4mm bubbles with an initial center-to-center separation of 12mm (3d). Vertical harmonic excitation is imposed at a frequency of 20Hz and an amplitude of 2mm, yielding a dimensionless vibration acceleration of $\Gamma \approx 3.22$. The continuous phase is described by a regularized Herschel-Bulkley constitutive model over the ranges of flow behavior index ($n = 0.1-0.8$), consistency index ($k = 0.5-4.0 \text{ Pa}\cdot\text{s}^n$), and yield stress ($\tau_0 = 0-20 \text{ Pa}$). The numerical framework is validated against benchmark data for bubble rise in a Bingham fluid, yielding an average deviation of 12.2% in the bubble vertical trajectory. The results show that increasing the consistency index and yield stress systematically increases hydrodynamic resistance and reduces bubble mobility. Increasing the flow behavior index also reduces the shear-thinning effect and, within the investigated range, leads to strongly mobility-limited conditions at high n . The trailing bubble generally exhibits a higher Reynolds number and lower drag coefficient than the leading bubble, indicating persistent hydrodynamic interaction between the bubbles. Fourier analysis of the bubble velocity shows that the phase difference between the bubble response and the imposed excitation changes systematically with rheological properties, reaching 88.3° under an intermediate rheological condition. Because the simulations are performed at a single forcing frequency, these phase results are interpreted as a frequency-specific dynamic response rather than as definitive evidence of resonance or anti-resonance. The results provide mechanistic insight into the coupled effects of rheology, bubble interaction, and external vibration, with relevance to vibration-assisted degassing of yield-stress fluids.

Keywords: Vibrational Degassing; Viscoplastic Fluid; Herschel–Bulkley Model; Bubble Dynamics; Open FOAM.

1. Introduction

Elimination of entrapped gas bubbles in highly viscous and yield stress fluids is an important need in polymer processing, energetic material manufacturing, pharmaceutical formulations, and other processes involving rheologically complex materials [1,2]. In such systems, the high viscosity and yield stress can strongly resist bubble migration and even immobilize the bubbles when the buoyancy-induced stresses are not enough to yield the surrounding material [3,4]. Thus, vibration-assisted degassing has been developed as a nonintrusive method to improve bubble mobility by modifying the local stress and shear rate fields without adding any mechanical parts to the fluid [5]. Likewise, resonant acoustic

mixing applies an oscillatory excitation to process highly viscous materials [6].

The dynamics of bubbles in yield-stress fluids are governed by the competition among buoyancy, capillary forces, viscous resistance, and the yield stress of the continuous phase. Unlike Newtonian fluids, bubble migration requires yielding of the material surrounding the gas-liquid interface, resulting in a localized yielded region, while the bulk fluid may remain unyielded [3, 4]. As shown by Pourzahedi et al. [7], the critical balance between buoyancy-induced stresses and yield resistance that controls the onset of bubble motion, demonstrates the inadequacy of the conventional Reynolds-number based descriptions for yield-stress fluids. Daneshi et al. [8] also demonstrated that bubble growth can alter the local stress field and cause yielding. These results indicate that the local rheological state plays an important role in bubble stability and mobility. The presence of neighboring bubbles leads to additional hydrodynamic and rheological interactions. Chaparian and Frigaard [9] have shown that interactions within bubble clouds can alter the regions yielded and lead to the formation of collective structures. In particular, Kordalis et al. [10] showed that two coaxially aligned bubbles in an elastoviscoplastic material can form a stress bridge between the leading and trailing bubbles, which promotes the latter to approach. The interaction is highly dependent on the initial separation and material properties, such as viscosity, elasticity, shear thinning, and yield stress. Kordalis et al. [11] also showed that variations in relative bubble size can lead to different interaction regimes, such as approach, separation, and an approximately constant equilibrium distance. The studies show that the behavior of a pair of bubbles cannot be predicted from that of isolated bubbles. However, their main concern was buoyancy-driven motion, and they did not consider the additional time-dependent stress field produced by external vibration.

In numerical simulation of such systems, accurate representation of the deformable gas-liquid interface is also needed. Fakhari and Fernandes [12] demonstrated that a geometric Volume of Fluid approach can accurately capture bubble deformation and rising dynamics in shear-thinning and elastoviscoplastic fluids, thus providing a suitable numerical basis for resolving transient multiphase flow in complex rheological media. External vibration can affect bubble transport through local strain rate, apparent viscosity, bubble deformation, and hydrodynamic drag. Zhan et al. [5] showed that controlled vibration can improve gas removal from

highly viscous non-Newtonian fluids, especially when the oscillatory shear decreases the apparent viscosity of shear-thinning materials. However, excessive excitation may promote gas re-entrapment. More specifically, Pavarpoor et al. [13] numerically investigated the removal of bubbles from a viscoplastic fluid. They established a correlation to determine appropriate degassing process parameters based on rheological and operating conditions. Their results demonstrated that the competition between buoyancy and viscoplastic resistance is a strong driver of bubble removal and established a direct relationship between the fluid's rheological properties and the vibration conditions required for effective degassing. This study thus provides an important process-level basis for vibration-assisted degassing of yield-stress fluids. However, the analysis was mainly focused on the removal of a single bubble and the determination of process parameters, rather than on the transient interactions and frequency response of multiple bubbles. Oscillatory forcing may also result in substantial changes in bubble motion and deformation. Bie et al. [14] reported significant variations of bubble trajectories under oscillatory flow. Kononov et al. [15] showed the importance of the vibration frequency and resonance for the dynamics of bubbles and drops in viscous fluids. Riccio and De Corato [16] also showed that periodic pressure oscillations can cause large strain rates around a bubble in a shear-thinning fluid, leading to substantial shear thinning and a large increase in the period-averaged rising velocity. These studies show that oscillatory forcing can strongly modify bubble dynamics. However, most have focused on isolated bubbles or fluids without a yield stress [6]. Thus, although there has been substantial progress in understanding bubble motion in yield-stress fluids, bubble-bubble interactions and vibration-assisted degassing remain poorly understood. In particular, the contribution of the Herschel-Bulkley constitutive parameters, including the flow behavior index (n), consistency index (k), and yield stress (τ_0), to the transient interaction and dynamic response of vertically aligned bubbles under forced vibration has not been systematically quantified. Previous studies of interacting bubbles have primarily considered quiescent or buoyancy-driven conditions [9,10,11], whereas vibration-assisted studies have mainly addressed individual bubble removal or process-level degassing [5,13]. The present study addresses this gap through a systematic numerical investigation of an in-line pair of equal-sized gas bubbles rising in a Herschel-Bulkley fluid under sinusoidal vertical vibration. The transient gas-liquid interface and bubble deformation are captured using the Volume of Fluid method within the OpenFOAM framework. The effects of n , k , and τ_0 are evaluated using bubble trajectories, bubble velocity, hydrodynamic dimensionless parameters, and the phase relationship between bubble motion and the imposed excitation. The objective is to clarify how rheological resistance modifies both the mean bubble mobility and the frequency-specific dynamic response of the interacting bubble pair.

2. Mathematical Formulation and Numerical Methodology

2.1. Volume of Fluid Method

The Transient two-phase flow of the gas bubbles and the surrounding viscoplastic liquid is simulated by the Eulerian Volume of Fluid (VOF) method. The VOF formulation uses a single set of conservation equations for the two immiscible phases and the gas-liquid interface is tracked by a phase fraction variable. In the present formulation, α is the volume fraction of the liquid phase in a computational cell. Accordingly, $\alpha=1$ corresponds to a liquid filled cell, $\alpha=0$ to a gas cell and $0<\alpha<1$ to an interfacial cell. The local mixture density and the dynamic viscosity are

evaluated as volume weighted quantities according to [17,18]:

$$M = \alpha M_L + (1 - \alpha) M_g \quad (1)$$

The interface moves with the flow. To preserve the interface, the transport equation for the interface must be solved. The interface transport equation is defined as follows:

$$\frac{\partial \alpha}{\partial t} + \bar{U} \cdot \nabla \alpha = 0 \quad (2)$$

The temporal evolution of the phase fraction is obtained from the conservative advection equation:

$$\frac{\partial \alpha}{\partial t} = \bar{U} \cdot \nabla \alpha + \nabla \cdot [\bar{U}_r \alpha (1 - \alpha)] \quad (3)$$

where U is the velocity field and U_r is the artificial compression velocity directed approximately normal to the interface. The compression term is activated only in interfacial cells and is introduced to reduce numerical diffusion of the gas-liquid interface. This formulation is consistent with the interface compression treatment implemented in *OpenFOAM* based on a counter-gradient flux correction [18,19]. The interface compression coefficient is set to $C_\alpha=1$ throughout the simulations. This value provides a suitable compromise between interface sharpness and numerical boundedness. Excessive compression can introduce mesh-oriented distortions and parasitic currents, whereas insufficient compression may lead to excessive numerical diffusion of the interface. The selected value is also consistent with the standard *OpenFOAM* interface compression formulation, for which a coefficient of unity is generally suitable for maintaining interface integrity [18,19].

2.2. Governing Equations and Vibration Modeling

The gas-liquid system is considered incompressible and isothermal. Consequently, the mixture satisfies the continuity equation:

$$\nabla \cdot U = 0 \quad (4)$$

and the momentum equation:

$$\frac{\partial (\rho \bar{U})}{\partial t} + \nabla \cdot (\rho \bar{U} \bar{U}) = -\nabla p + \nabla \cdot \tau + \rho \bar{g} + F_s + F_v \quad (5)$$

ρ is the local mixture density, U is the velocity vector, p is the modified pressure, g is the gravitational acceleration vector, τ is the deviatoric stress tensor, F_v is the vibration-induced body force and the last term on the right-hand side represents the effect of surface tension between the two phases. The vertical motion of the vessel is prescribed as a sinusoidal displacement:

$$y(t) = A \sin(2\pi f t) \quad (6)$$

where A is the vibration amplitude and f is the excitation frequency. Rather than deforming the computational mesh, the vibration is represented by an equivalent body force in a non-inertial reference frame attached to the vessel. The corresponding vertical acceleration of the vessel is:

$$a_v(t) = -A(2\pi f)^2 \sin(2\pi f t) \quad (7)$$

and the associated fictitious acceleration acting on the fluid has the opposite sign. With the vertical unit vector denoted by j , the

vibration contribution used in the present formulation is therefore expressed as:

$$F_v = \rho A (2\pi f)^2 \sin(2\pi ft) j \quad (8)$$

where the sign corresponds to the selected coordinate convention. This formulation enables the global vibrational excitation of the vessel to be reproduced without explicitly solving a moving mesh problem. The use of a body force to represent vibration is particularly advantageous for a fixed computational domain because the prescribed excitation can be introduced directly into the momentum equation while maintaining a stationary mesh [20,21].

The interfacial surface tension force is represented using the Continuum Surface Force (CSF) formulation [22]:

$$F_s = \sigma \kappa \nabla \alpha \quad (9)$$

where σ is the gas-liquid surface tension coefficient and κ is the local interface curvature. The curvature is evaluated from the divergence of the interface normal vector as:

$$\kappa = -\nabla \cdot \left(\frac{\nabla \alpha}{|\nabla \alpha|} \right) \quad (10)$$

The CSF formulation converts the interfacial tension force into a volumetric force concentrated within the numerically resolved interface and is therefore compatible with the VOF framework [22]. The accurate representation of this force is particularly important for the present problem because surface tension contributes directly to bubble deformation and consequently affects the instantaneous hydrodynamic response under vibration.

2.3. Herschel-Bulkley Rheological Model

The continuous liquid phase is modeled as an incompressible Herschel-Bulkley fluid exhibiting yield-stress and shear-thinning behavior. The constitutive relation is written in terms of the deviatoric stress tensor as:

$$\tau = 2\eta(\dot{\gamma})D \quad (11)$$

where D is the rate of deformation tensor,

$$D = \frac{1}{2} [\nabla U + (\nabla U)^T] \quad (12)$$

and $\dot{\gamma}$ is the second invariant of the rate of deformation tensor. The Herschel-Bulkley model is characterized by three constitutive parameters: the yield stress τ_0 , consistency index k , and flow behavior index n . For the shear-thinning fluids considered in this study, $0 < n < 1$. The corresponding apparent viscosity can be expressed as:

$$\eta(\dot{\gamma}) = k\dot{\gamma}^{n-1} + \frac{\tau_0}{\dot{\gamma}} \quad (13)$$

for $\dot{\gamma} > 0$. The first term represents the shear-dependent contribution of the power-law component, whereas the second term accounts for the yield stress contribution [23,24]. Direct implementation of the Herschel-Bulkley equation produces a mathematical singularity as $\dot{\gamma} \rightarrow 0$, because the apparent viscosity tends toward

infinity in the unyielded region. Therefore, the Papanastasiou regularization is employed to obtain a numerically continuous constitutive relation [24]:

$$\eta(\dot{\gamma}) = k\dot{\gamma}^{n-1} + \frac{\tau_0}{\dot{\gamma}} [1 - \exp(-m\dot{\gamma})] \quad (14)$$

where m is the regularization parameter. In the limit of sufficiently large m , the regularized formulation approaches the original Herschel-Bulkley constitutive behavior while avoiding the singular viscosity at zero strain rate. The regularization parameter used in the present simulations is $m=10^3$ s. A maximum viscosity of 10^3 Pa.s is additionally imposed to prevent excessive ill-conditioning of the momentum equation in nearly unyielded regions. The influence of the regularization parameter should be interpreted as a numerical approximation of the unyielded zones rather than as a modification of the underlying physical constitutive model [24,25]. The generalized Reynolds number is used to characterize the relative importance of inertial and rheological effects during bubble motion. For the Herschel-Bulkley fluid, the Reynolds number adopted in this study is defined as:

$$Re = \frac{\rho_L U_b^{2-n} d_e^n}{k + \tau_0 (d_e / U_b)^n} \quad (15)$$

where U_b is the instantaneous vertical velocity of the bubble and d_e

is its equivalent spherical diameter ($d_{eq}(t) = \sqrt{\frac{4A_b(t)}{\pi}}$). The

Eötvös number is defined as:

$$Eo = \frac{(\rho_L - \rho_G) g d_e^2}{\sigma} \quad (16)$$

and represents the relative importance of buoyancy and surface tension. These dimensionless groups provide a convenient basis for interpreting the competition among inertia, yield stress, viscous resistance, buoyancy, and interfacial tension during the bubble ascent process [23,26].

2.4. Frequency Domain Analysis and Phase Response

Because the principal objective of the present study is not limited to the mean bubble rise velocity but also includes the dynamic coupling between bubble motion and the imposed vibration, the instantaneous bubble velocity is analyzed in both the time and frequency domains.

It should be noted that in heavily non-linear yield-stress fluids, the effective stiffness and damping of the surrounding medium are governed by the instantaneously yielded regions. Therefore, a constant natural frequency as defined in classical linear oscillatory systems cannot be explicitly determined.

Accordingly, the present phase analysis rigorously evaluates the frequency-specific dynamic response at the imposed 20 Hz excitation rather than comparing it against an elusive natural frequency.

The vertical position of each bubble is obtained from the time-dependent centroid of the gas phase, and the corresponding instantaneous velocity is calculated from the temporal evolution of

the bubble centroid. For each bubble, the resulting velocity signal is represented as:

$$U_b(t_j), \quad j = 0, 1, \dots, N-1 \quad (17)$$

where N is the number of uniformly sampled temporal data points. Before applying the Fourier transform, the mean component is removed according to:

$$u_{b'}(t_j) = U_b(t_j) - \bar{U}_b \quad (18)$$

Where:

$$\bar{U}_b = \frac{1}{N} \sum_{j=0}^{N-1} U_b(t_j) \quad (19)$$

Represents the mean bubble rise velocity. Removal of the mean component prevents the zero-frequency contribution associated with the net upward migration of the bubble from dominating the dynamic spectrum. Fourier-based analysis of bubble trajectories and velocities has previously been employed to distinguish forced oscillations from the mean translational motion and to identify dominant frequencies in bubble dynamics [27,28]. The spectral analysis was performed over a stationary time window after the initial transient startup phase had completely decayed. A standard rectangular windowing function was applied to the detrended velocity signals. This specific windowing choice was made to preserve the absolute amplitude accuracy of the velocity fluctuations prior to executing the discrete Fourier transform. The discrete Fourier transform of the fluctuating bubble velocity is then calculated as:

$$\hat{U}_b(f_k) = \sum_{j=0}^{N-1} u_{b'}(t_j) \exp\left(-i \frac{2\pi k j}{N}\right) \quad (20)$$

Where:

$$f_k = \frac{k}{N\Delta t}, \quad k = 0, 1, \dots, N-1 \quad (21)$$

is the discrete frequency associated with the k -th Fourier coefficient and Δt is the temporal sampling interval. The corresponding frequency resolution is:

$$\Delta f = \frac{1}{N\Delta t} = \frac{1}{T} \quad (22)$$

where $T = N\Delta t$ is the total duration of the analyzed signal. The single-sided amplitude spectrum is obtained from the magnitude of the Fourier coefficients. The dominant response frequency is identified from the frequency corresponding to the maximum spectral amplitude after excluding the zero-frequency component. To quantify the phase coupling between the imposed vibration and the bubble response, the vertical vibration acceleration is used as the reference excitation signal:

$$a_v(t) = A(2\pi f)^2 \sin(2\pi ft) \quad (23)$$

Its Fourier transform is denoted by $\hat{A}_v(f_k)$. The phase of the bubble response and the excitation signal are calculated from:

$$\begin{aligned} \phi_b(f_k) &= \arg[\hat{U}_b(f_k)] \\ \phi_v(f_k) &= \arg[\hat{A}_v(f_k)] \end{aligned} \quad (24)$$

respectively. The phase lag of the bubble velocity relative to the imposed vibration is then obtained from:

$$\Delta\phi_b(f_k) = \phi_b(f_k) - \phi_v(f_k) \quad (25)$$

with the phase difference wrapped to the interval $[-180, 180]$. The phase lag is evaluated at the excitation frequency f . A value of $\Delta\phi_b = 0^\circ$ indicates an in-phase response, whereas positive or negative values indicate phase lead or lag. For a quantitative measure of the dynamic response, the normalized amplitude ratio at the forcing frequency is defined as:

$$H_b(f) = \frac{|\hat{U}_b(f)|}{|\hat{A}_v(f)|} \quad (26)$$

which characterizes the velocity response of the bubble relative to the imposed oscillatory acceleration. The corresponding complex frequency response may be expressed as:

$$H_b(f) = \frac{\hat{U}_b(f)}{\hat{A}_v(f)} = |H_b(f)| \exp[i\Delta\phi_b(f)] \quad (27)$$

Thus, the magnitude of H_b quantifies the strength of the dynamic coupling, whereas $\Delta\phi_b$ quantifies the phase relationship. This provides a direct framework for distinguishing strongly coupled, weakly coupled, and phase-shifted bubble responses [29,30].

3. Computational Setup and Solution Strategy

3.1. Computational Domain and Physical Configuration

The numerical simulations are performed in a two-dimensional rectangular computational domain. The two-dimensional formulation is adopted to maintain a computationally tractable framework for the systematic parametric investigation of the Herschel-Bulkley constitutive parameters while retaining direct resolution of the gas-liquid interface. This approximation represents the bubbles as cylindrical structures in the computational plane and therefore does not reproduce inherently three-dimensional phenomena such as azimuthal wake structures, out-of-plane deformation, or three-dimensional vortex shedding. Accordingly, the absolute values of hydrodynamic quantities should be interpreted within the limitations of two-dimensional flow. The primary objective of the simulations is therefore to resolve the relative changes in bubble motion, bubble-bubble interaction, and frequency response produced by variations in n , k , τ_0 and the imposed vibration. Such a distinction between quantitative prediction and comparative mechanistic analysis is important when interpreting two-dimensional multiphase simulations [26,30]. The computational domain has dimensions of 100mm in width and 100mm in height. Initially, the lower 70% of the domain is filled with the viscoplastic liquid, whereas the upper 30% is occupied by air at atmospheric pressure. Two identical gas bubbles with an equivalent diameter of $d=4$ mm are initially positioned along the central vertical axis of the domain. Their initial center-to-center distance is 12mm, corresponding to $3d$, only the coaxially aligned configuration with an initial center-to-center separation of $3d$ was considered in the present study. No separate simulations were performed for a $2.5d$ separation. The selected configuration was used to represent longitudinal bubble-bubble interaction while maintaining a finite initial separation between the

two interfaces, and imposed excitation setup are illustrated in Figure 1.

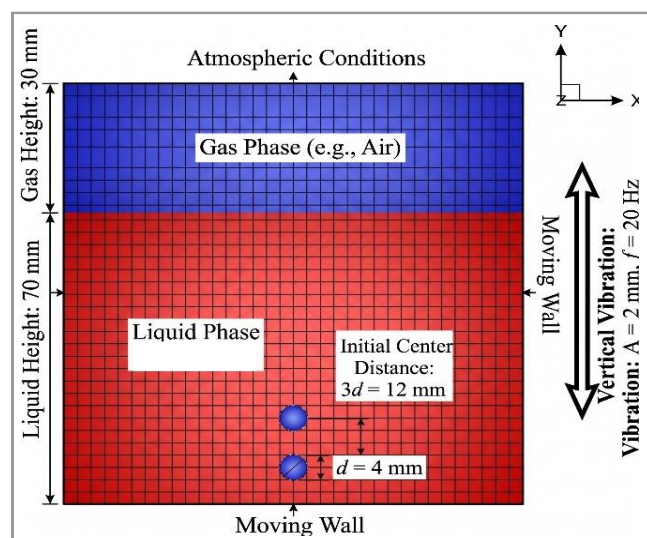


Fig 1. View of the geometry of the vibrational degassing problem.

The vessel is subjected to a sinusoidal vibration with a frequency of $f=20\text{Hz}$ and an amplitude of $A=2\text{mm}$. For the present simulations, the angular frequency is $\omega = 2\pi f = 125.66 \text{ rad s}^{-1}$. The corresponding velocity and acceleration amplitudes are $A\omega = 0.251 \text{ m s}^{-1}$ and $A\omega^2 = 31.58 \text{ m s}^{-2}$, respectively. The dimensionless vibration acceleration is therefore $\Gamma = A\omega^2/g = 3.22$. The vibration is applied in the vertical direction. The numerical simulations are performed in a fixed computational domain using the equivalent non-inertial body-force formulation. The transient simulations were conducted continuously until the upper bubble reached the gas-liquid free surface. Consequently, the total simulation duration and the corresponding number of experienced vibration cycles were not strictly fixed, but rather increased proportionally with the ascent time, which varied depending on the specific rheological resistance of each case. This approach ensured that a sufficient number of cycles were captured for all cases to establish a stationary oscillatory state prior to phase analysis.

3.2. Boundary Conditions

No-slip velocity conditions (*noSlip*) are imposed on the bottom and lateral solid walls. A zero-gradient condition is specified for the phase fraction (α) at the solid boundaries, while the modified pressure (p_{rgh}) is treated using the corresponding zero-gradient condition. These conditions prevent artificial tangential motion of the liquid at the solid walls and provide a closed lateral confinement for the two-bubble trajectories.

The upper boundary is treated as an open atmospheric boundary to permit the escape of the gas phase after bubble migration through the liquid free surface. A total pressure condition of 0 Pa (*totalPressure*) is imposed for p_{rgh} at the top boundary. The velocity field is treated using the *pressureInletOutletVelocity* condition, while the phase fraction is specified using an *inletOutlet* condition. These mixed open boundary conditions allow gas to leave the computational domain while limiting unphysical inflow of the liquid phase [18,20].

3.3. Numerical Solver and Discretization

The two-phase flow equations are solved using the *interFoam* solver available in *OpenFOAM* version 9, utilizing a VOF-based phase fraction formulation and a single mixture momentum equation [18,20]. The Herschel-Bulkley rheological model is natively implemented within the standard *OpenFOAM* transport libraries, enabling the *interFoam* solver to seamlessly accommodate the yield-stress and shear-thinning characteristics of the continuous phase without requiring external modifications. The spatial and temporal discretization schemes are selected to provide an appropriate balance between numerical stability, interface boundedness, and temporal accuracy. The temporal derivative is discretized using the second-order backward *Euler* scheme available in *OpenFOAM*, which strongly mitigates the phase-lag errors typically associated with first-order methods. Spatial gradients are evaluated using the second-order *Gauss linear* scheme. The convective terms in the momentum equation are discretized using the *Gauss linearUpwind* scheme, while the phase fraction convection is treated using the bounded *Gauss vanLeer* formulation combined with interface compression [18,19]. The pressure-velocity coupling is performed using the *PIMPLE* algorithm. Three outer *PIMPLE* correction loops and two inner pressure correction iterations are employed within each time step. The time step is dynamically adjusted according to the local Courant number, with the maximum Courant number maintained below 0.25 and the interface Courant number maintained below 0.1. These stringent restrictions provide adequate temporal resolution of the strongly transient bubble motion and perfectly capture the phase relationship with the periodic forcing.

4. Results and Discussion

4.1. Grid Independence Assessment

A comprehensive grid independence study was conducted to ensure the spatial convergence of the numerical solution and to eliminate grid-induced numerical diffusion, which is critical for accurate interface tracking in the Volume of Fluid (VOF) method. In viscoplastic multiphase flows, an inadequate grid resolution can smear the gas-liquid interface and artificially modify the local yield stress field, whereas an excessively fine grid incurs prohibitive computational costs, particularly for transient frequency-domain analyses.

To evaluate spatial convergence, three uniform structured meshes with increasing densities coarse (100×100), medium (400×400), and fine (800×800) were systematically tested. The grid independence analysis was performed based on a benchmark laboratory configuration matching the dimensions ($192 \times 80 \text{ mm}$) and fluid properties (Bingham plastic) of the validation baseline. The primary parameter used for this assessment was the time-resolved vertical position of the bubble centroid.

To establish the adequacy of the spatial resolution, the "Cells Per Bubble Diameter" (CPBD) metric was evaluated. For the medium mesh density (400×400), the spatial resolution corresponds to approximately 16 computational cells across the initial equivalent diameter of the bubble ($CPBD \approx 16$). For the medium mesh, the initial bubble diameter is resolved by approximately 16 cells, providing a spatial resolution considered adequate for the present VOF calculations and representing a practical compromise between interface resolution and computational cost.

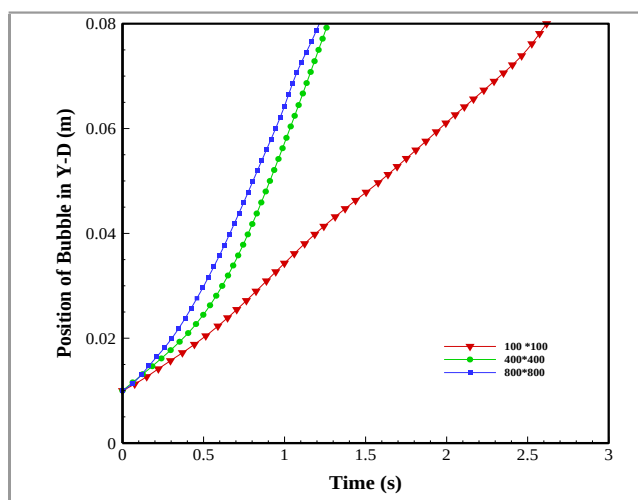


Fig 2. Variation of Bubble Position Over Time for Grid Independence Analysis.

Figure 2 presents the temporal variation of the bubble position for the three evaluated mesh resolutions. As depicted, the coarse grid fails to accurately capture the ascent dynamics due to high numerical dissipation. In contrast, the trajectories predicted by the medium and fine meshes exhibit asymptotic convergence. Reviewing the quantitative deviations, a maximum instantaneous discrepancy of approximately 13% is observed between the medium and fine grids during the initial acceleration and extreme deformation phase of the bubble. However, this transient peak error rapidly diminishes as the flow develops, yielding a time-averaged spatial deviation of less than 4.5% over the complete ascent trajectory. The medium mesh was therefore selected as a practical computational compromise for the subsequent parametric simulations. Although the maximum instantaneous difference between the medium and fine meshes reaches approximately 13% during the early deformation stage, the time-averaged trajectory deviation remains below 4.5%. The selected mesh is consequently considered adequate for comparative analysis of the rheological effects investigated in this study, while the reported instantaneous discrepancy should be kept in mind when interpreting absolute transient quantities.

4.2. Validation of the Numerical Model

To assess the numerical framework, the present model was compared with published benchmark data for bubble rise in a viscoplastic fluid reported by Pavarpoor et al. [13]. This comparison provides a validation of the numerical treatment of bubble motion and viscoplastic resistance under the vibration conditions. Because the published benchmark does not reproduce the complete configuration of the present study, including the vertically aligned bubble pair and imposed vibration, the comparison is considered a validation of the underlying bubble-rise and rheological treatment rather than a direct validation of the full coupled vibration-driven two-bubble system. A quantitative comparison between the present numerical simulation and the experimental benchmark is provided in Figure 3. The graph tracks the smoothed vertical position of the bubble over a time period of approximately 38 seconds. As depicted, the CFD model successfully reproduces the macroscopic ascent trajectory, effectively capturing the non-linear, slow-kinematic behavior dictated by the unyielded regions and apparent viscosity of the surrounding viscoplastic fluid. The simulated vertical positions demonstrate strong quantitative agreement with the experimental data across the entire evaluation period. A minor transient discrepancy is observed during the intermediate ascent phase (particularly between $t = 15$ s and $t = 25$ s), where the simulation slightly underpredicts the bubble position. However, the maximum relative deviation remains strictly bounded within acceptable engineering limits for multiphase non-Newtonian flows. This localized deviation is a direct and expected consequence of the two-dimensional geometric approximation applied in the numerical domain.

4.3. Analysis of Parameters Affecting the Rise of Bubble Pairs Under Vibration

To evaluate the fundamental role of rheological properties during vibration-assisted degassing, a systematic parametric study was conducted. The baseline geometry and boundary conditions correspond to the configuration detailed in Section 3.1. The Herschel-Bulkley constitutive parameters, flow behavior index (n), consistency index (k), and yield stress (τ_0), were varied independently to isolate their specific hydrodynamic and dynamic effects on the bubble pair. Table 1 summarizes the range of rheological parameters employed in these simulations.

Table 1. Rheological Properties and Parameters of Fluids Used in the Present Study.

Fluid type	Yield stress τ_0 [Pa]	Density ρ [kg/m^3]	Consistency index k [Pa.s ⁿ]	Flow behavior index n (-)	Surface tension σ [N/m]
Herschel-Bulkley	0-1-10-20	1000	0.5-1-2-4	0.1-0.2-0.4-0.8	0.034
Newtonian (air)	-	1.25	0.00001789	1	-

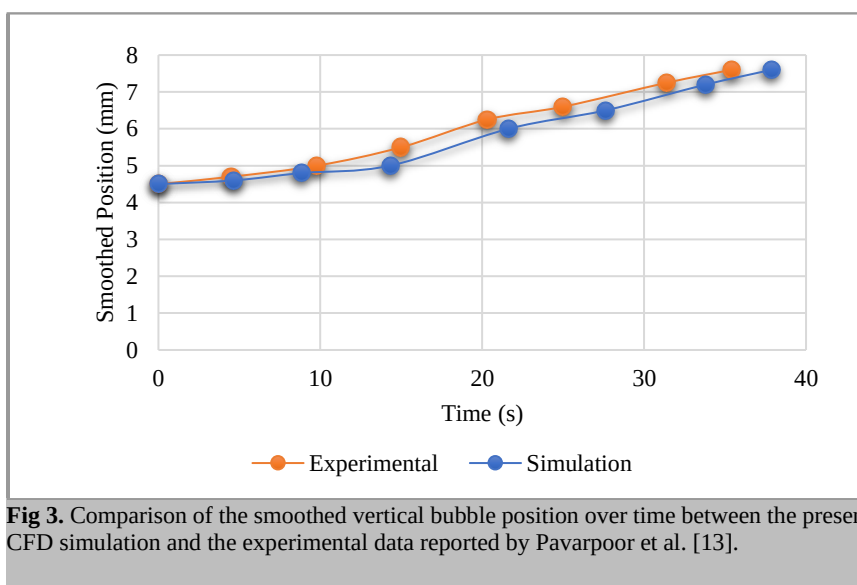


Fig 3. Comparison of the smoothed vertical bubble position over time between the present CFD simulation and the experimental data reported by Pavarpoor et al. [13].

4.3.1. Effect of the Flow Behavior Index (n)

Hydrodynamic Analysis

The flow behavior index (n) dictates the degree of shear-thinning in the non-Newtonian fluid. As n approaches unity, the fluid progressively loses its shear-thinning characteristics, resulting in higher apparent viscosity at equivalent shear rates. Figure 4 illustrates the transient vertical trajectories of the upper and lower bubbles under continuous vibration for various flow behavior indices. The corresponding volume fraction contours, illustrating the interface morphology and bubble separation over time, are presented in Figure 5.

The kinematic response of the bubble pair is highly sensitive to n . At highly shear-thinning conditions ($n=0.1$ and $n=0.2$),

the imposed vibration effectively reduces the local apparent viscosity, allowing both bubbles to ascend rapidly. The time-averaged dimensionless hydrodynamic parameters, summarized in Table 2, provide quantitative insight into this regime. For $n=0.1$, the lower bubble exhibits a generalized Reynolds number (Re) of 0.982 and a drag coefficient (C_D) of 24.45, whereas the upper bubble experiences greater resistance ($Re=0.556$, $C_D=43.26$). This velocity differential arises from a wake-mediated interaction. The lower bubble migrates through the sheared, yielded wake generated by the leading upper bubble. The localized reduction in effective viscosity in this disturbed region significantly lowers the drag acting on the trailing bubble, accelerating its ascent.

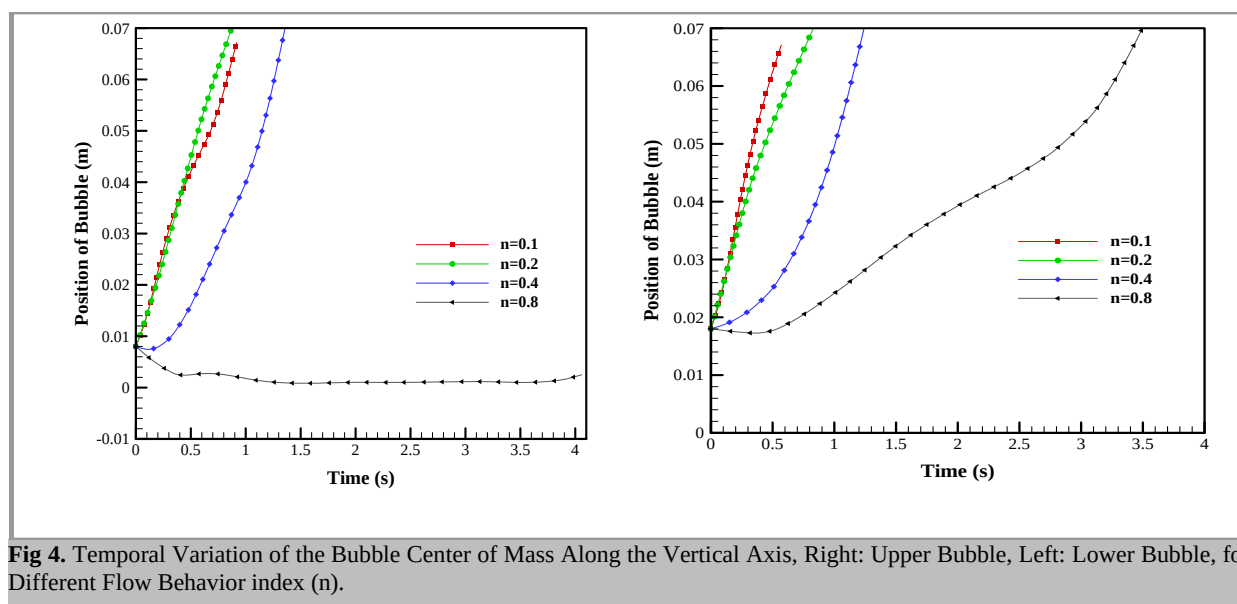


Fig 4. Temporal Variation of the Bubble Center of Mass Along the Vertical Axis, Right: Upper Bubble, Left: Lower Bubble, for Different Flow Behavior index (n).

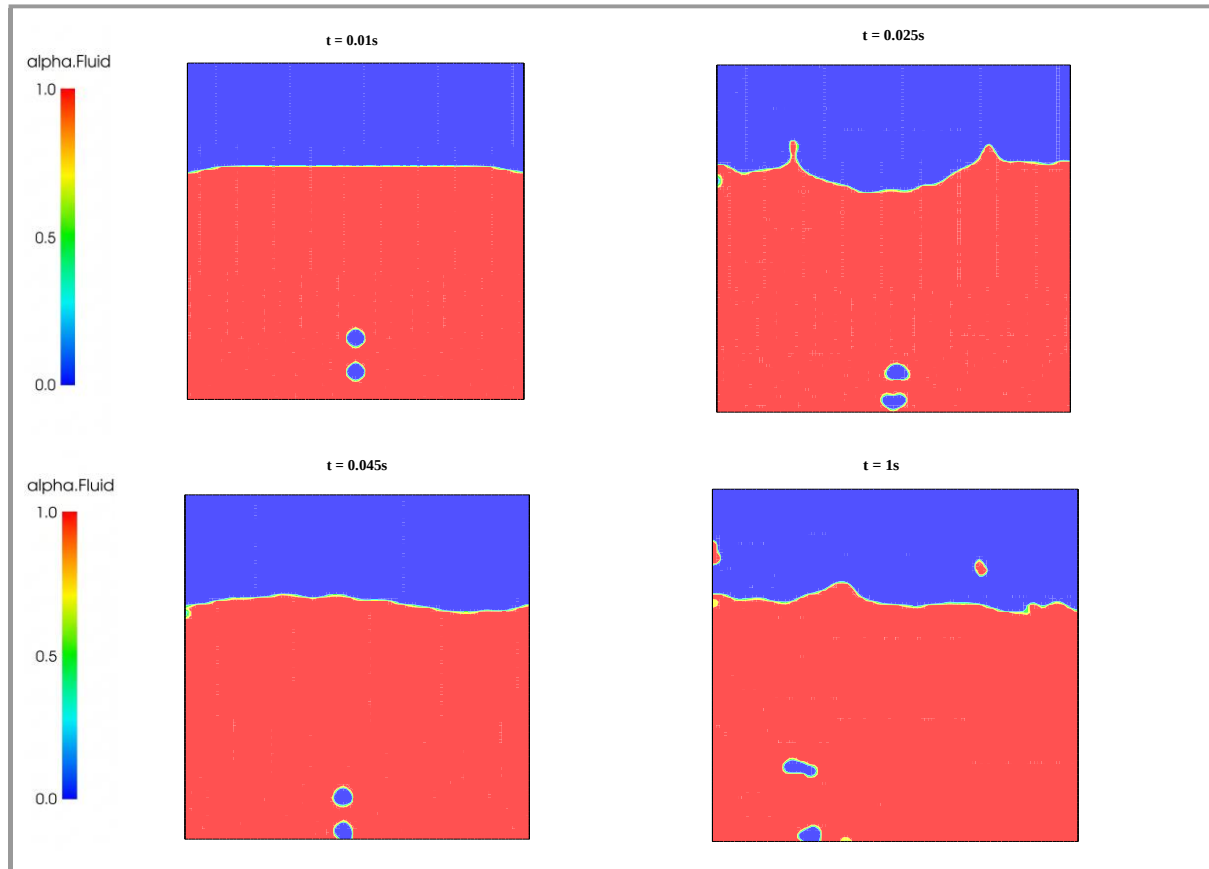


Fig 5. Bubble volume fraction contour at different times for $n=0.8$.

As n increases to 0.4, the shear-thinning mechanism weakens. The unyielded and highly viscous zones become more resistant to the vibration-induced strain rates. Consequently, a sharp hydrodynamic transition occurs. Table 2 demonstrates that the Reynolds number for the lower bubble drops significantly from 0.861 (at $n=0.2$) to 0.136, accompanied by a massive increase in the drag coefficient to 133.57. A similar deceleration is observed for the upper bubble ($Re=0.122$, $C_D=196.52$). At this stage, the fluid approaches a transitional balance between its flowability and viscoplastic resistance.

Table 2. Average dimensionless numbers for the upper and lower bubbles at different Flow Behavior Index.

Flow Behavior Index	Bubble Location	Eo	$\overline{Re}$	$\overline{C}_D$
0.1	up	722.23	0.556	43.261
	Bottom	777.78	0.982	24.45
0.2	up	733.35	0.493	48.78
	Bottom	759.26	0.861	27.89
0.4	up	751.85	0.122	196.523
	Bottom	737.04	0.136	133.57
0.8	up	764.81	0.133	1729.58
	Bottom	721.35	0	∞

When the flow behavior index is further increased to $n=0.8$, the fluid exhibits near-rigid behavior outside the immediate high-shear zones. The high apparent viscosity severely restricts momentum transfer. As indicated in Table 2, the lower bubble becomes completely arrested ($Re \approx 0$, $C_D \rightarrow \infty$) due to the inability of buoyancy and local vibration stresses to overcome the surrounding yield envelope. The upper bubble, benefiting slightly from the proximity to the free surface and local instabilities, exhibits minimal mobility ($Re=0.133$, $C_D=1729.58$) but is heavily damped. The Eötvös numbers (Eo) remain relatively stable across all n values (ranging between 721 and 777), indicating that the macroscopic shape of the bubbles is governed primarily by surface tension and yield stress constraints rather than inertial deformation.

Frequency and Phase Response Analysis

To accurately identify the occurrence of dynamic resonance, the phase difference ($\Delta\phi$) between the imposed vessel vibration and the instantaneous velocity of the bubbles was extracted using Fast Fourier Transform (*FFT*). Table 3 presents the evaluated phase shifts and the corresponding dynamic states for the bubble pair.

The phase differences listed in Table 3 describe the relative timing between the imposed vibration and the bubble response. These phase differences alone are not sufficient to establish resonance or anti-resonance.

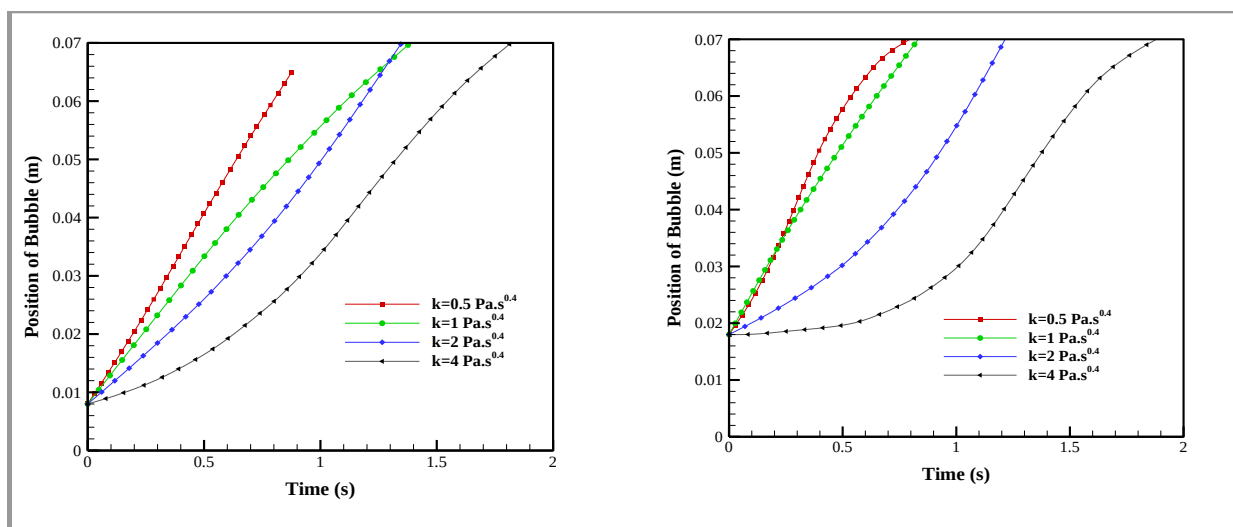


Fig 6. Temporal variations of the center of mass for the (right) upper bubble and (left) lower bubble at different values of k .

Table 3. Evaluation of phase difference and vibration-induced resonance across different flow behavior indices.

Flow behavior index	Bubble Location	Phase Difference	Qualitative phase-response description
0.1	up	32	Relatively small phase lag
	Bottom	28	Relatively small phase lag
0.2	up	53.5	Intermediate phase lag
	Bottom	46	Intermediate phase lag
0.4	up	88.3	Near-quadrature phase response
	Bottom	74.25	Intermediate phase lag
0.8	up	176	Strongly phase-shifted response
	Bottom	--	Strongly phase-shifted response

A definitive identification of resonance requires an amplitude-based analysis, preferably through a frequency sweep or a frequency-response curve. Accordingly, the terms “resonance” and “anti-resonance” are not used here as definitive physical classifications. The present results are described in terms of phase lag and phase-shifted bubble response.

In forced oscillatory systems, the phase lag serves as a critical indicator of energy transfer. At $n=0.1$ and $n=0.2$, the system exhibits relatively small phase lags (ranging from 28° to 53.5°). These values indicate a stiff, sub-resonant response where the fluid transmits the kinematic excitation with minimal dynamic delay, but the absorption of oscillatory energy by the bubbles remains suboptimal.

A critical shift occurs at $n=0.4$. The phase lag for the upper bubble reaches 88.3°, closely approximating the 90° quadrature condition that characterizes optimal phase coupling in driven oscillators. The lower bubble simultaneously approaches a near-resonant state ($\Delta\phi = 74.25^\circ$). This alignment represents a dynamic resonance state where the transfer of vibrational

energy to the fluid is maximized, effectively compensating for the increased apparent viscosity and maintaining bubble mobility despite the weakened shear-thinning behavior. Conversely, at $n=0.8$, the severe increase in viscosity induces excessive damping within the continuous phase. The phase lag of the upper bubble shifts dramatically to 176°. This condition constitutes an Strongly phase-shifted response state, where the kinematic response of the bubble is almost entirely out of phase with the periodic excitation. The externally applied force opposes the natural motion of the bubble, severely dissipating energy and contributing to the complete hydrodynamic arrest observed in the temporal trajectories.

4.3.2. Effect of the Consistency Index (k) Hydrodynamic Analysis

The consistency index (k) determines the magnitude of the viscous contribution to the Herschel–Bulkley constitutive response and therefore strongly influences the resistance of the continuous phase to bubble motion. At a fixed flow behavior index and yield stress, increasing k increases the apparent viscosity over the relevant shear-rate range and consequently reduces bubble mobility. Figure 6 presents the temporal variations of the vertical positions of the two bubbles for different values of k under identical vibrational excitation. At the lowest consistency index, $k=0.5 \text{ Pa}\cdot\text{s}^n$, the bubbles exhibit comparatively rapid upward migration. As shown in Table 4, the Reynolds numbers of the lower and upper bubbles are 0.449 and 0.351, respectively. The higher Reynolds number of the lower bubble is associated with its interaction with the disturbed flow generated by the leading bubble. In particular, the trailing bubble experiences a modified hydrodynamic environment within the wake of the upper bubble, which reduces its effective resistance relative to the isolated leading bubble. Consistent with this behavior, the drag coefficient of the lower bubble is 53.29, compared with 68.35 for the upper bubble.

The numerical values in Table 4 show that increasing the consistency index decreases the Reynolds number and increases the drag coefficient for both bubbles. Increasing the consistency index to $k=2 \text{ Pa}\cdot\text{s}^n$ produces a substantial reduction in bubble

mobility. The Reynolds numbers decrease to 0.122 and 0.136 for the upper and lower bubbles, respectively, while the corresponding drag coefficients increase to 196.52 and 133.57.

Table 4. Average dimensionless numbers for the upper and lower bubbles at different Consistency index.

Consistency index	Bubble Location	Eo	$\overline{Re}$	$\overline{C}_D$
0.5	up	685.18	0.351	68.353
	Bottom	720.37	0.449	53.29
1	up	714.815	0.272	88.308
	Bottom	731.48	0.382	62.673
2	up	751.852	0.122	196.523
	Bottom	737.037	0.136	133.57
4	up	759.259	0.0397	604.743
	Bottom	768.52	0.0531	449.73

This simultaneous decrease in Re and increase in C_D indicates that the increasing consistency of the continuous phase enhances the hydrodynamic resistance experienced by both bubbles. Nevertheless, the lower bubble continues to exhibit a comparatively smaller drag coefficient, reflecting the persistent influence of the wake generated by the leading bubble. A further increase to $k=4$ Pa.sⁿ produces an even stronger suppression of bubble motion. The Reynolds numbers decrease to 0.0397 for the upper bubble and 0.0531 for the lower bubble, whereas the corresponding drag coefficients increase to 604.74 and 449.73. The pronounced reduction in Re indicates that the bubble motion occurs in an increasingly viscously dominated regime. Simultaneously, the large increase in C_D demonstrates the substantial increase in hydrodynamic resistance associated with the higher consistency of the continuous phase. The lower bubble nevertheless maintains a smaller drag coefficient than the upper bubble, suggesting that the hydrodynamic modification of the flow field by the leading bubble remains important even under strong viscous resistance. The *Eötvös* number (Eo) is governed by the density difference, gravitational acceleration, surface tension, and characteristic bubble diameter. Accordingly, since the fluid properties are fixed, any minor variations in the evaluated Eo among the simulated cases originate solely from the transient changes in the instantaneous deformed equivalent diameter of the bubbles rather than from the variation of k itself. Overall, the hydrodynamic results demonstrate that increasing k systematically suppresses bubble mobility, decreases the Reynolds number, and increases the drag coefficient, while the relative difference between the two bubbles confirms the continuing role of wake-mediated hydrodynamic interaction.

Frequency and Phase Response Analysis

The influence of the consistency index on the dynamic response of the bubble pair was further examined using Fourier analysis. Table 5 summarizes the phase differences obtained from the Fourier spectra of the imposed wall excitation and the corresponding bubble velocity responses.

At $k=0.5$ and 1 Pa.sⁿ, the phase lags of the upper bubble are 59° and 77°, respectively. The progressive increase in phase lag indicates that increasing fluid consistency modifies the temporal response of the bubble relative to the imposed periodic excitation. At $k=2$ Pa.sⁿ, the phase lag reaches 88.3°, corresponding to an approximately quadrature responses. This near-quadrature condition indicates strong dynamic coupling between the imposed vibration and the bubble motion.

Accompanied by the observed kinematic response, this condition can be interpreted as a near-resonant operational state.

At $k=2$ Pa.sⁿ, the phase lag reaches 88.3°, corresponding to an approximately quadrature response. This near-quadrature condition indicates strong dynamic coupling between the imposed vibration and the bubble motion. Accompanied by the observed kinematic response, this condition can be interpreted as a near-resonant operational state.

Table 5. Evaluation of phase shifts and resonance phenomena resulting from vibration, across different fluid Consistency index.

Consistency index	Bubble Location	Phase Difference	Qualitative phase-response description
0.5	up	59	Relatively small phase lag
	Bottom	41	Relatively small phase lag
1	up	77	Intermediate phase lag
	Bottom	75	Intermediate phase lag
2	up	88.3	Near-quadrature phase response
	Bottom	74.25	Intermediate phase lag
4	up	121	Strongly phase-shifted response
	Bottom	198	Strongly phase-shifted response

At the highest consistency index, $k=4$ Pa.sⁿ, the phase lag increases to 121° for the upper bubble and 198° for the lower bubble. The substantial phase shift, together with the pronounced reduction in bubble mobility, indicates that the increase in fluid consistency strongly damps and desynchronizes the bubble response relative to the imposed excitation. The larger phase difference observed for the trailing bubble further suggests that the hydrodynamic interaction between the two bubbles contributes to the dynamic response under strongly viscous conditions. Overall, the combined hydrodynamic and frequency-domain results indicate that increasing k produces two coupled effects: it increases the resistance to bubble translation and progressively modifies the phase relationship between bubble motion and external excitation. The intermediate value of $k=2$ Pa.sⁿ exhibits the strongest near-quadrature coupling, whereas the highest value of $k=4$ Pa.sⁿ corresponds to a strongly damped and asynchronous response. These observations demonstrate that the consistency index does not merely control the mean bubble rise rate but also plays an important role in determining the dynamic response of the bubble pair to external vibration.

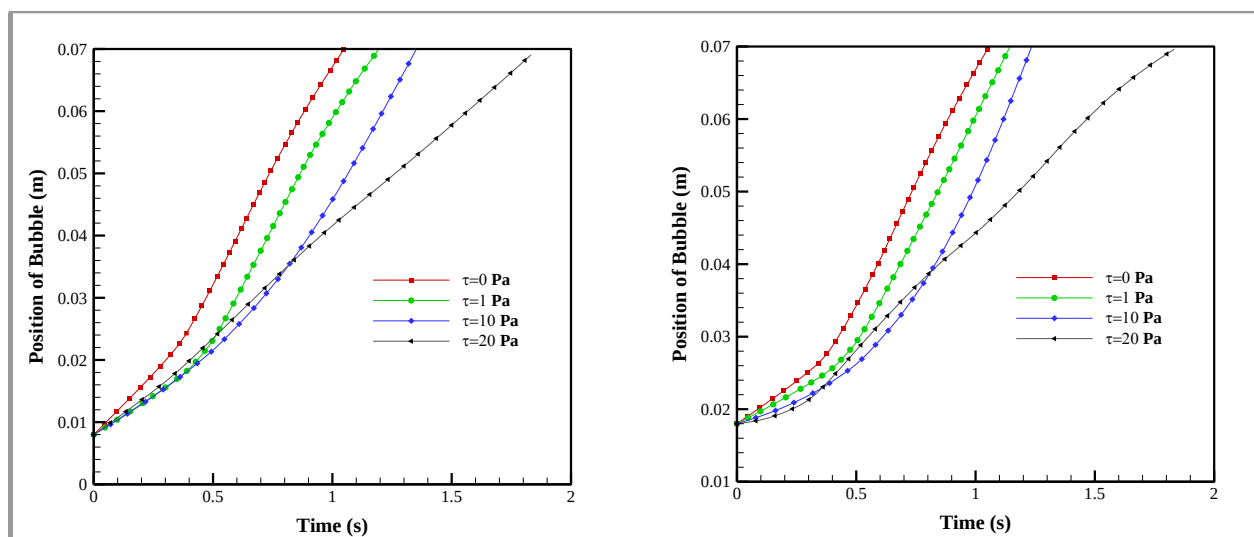


Fig 7. Time variations of the center of mass for the upper (right) and lower (left) bubbles at various yield stress values.

4.3.3. Effect of the Yield Stress (τ_0)

Hydrodynamic Analysis

The yield stress, τ_0 , represents the stress threshold that must be exceeded for irreversible deformation of the viscoplastic fluid to occur. As τ_0 increases, a larger portion of the surrounding fluid can remain in an effectively unyielded state, while deformation becomes increasingly localized in regions where the local stress exceeds the yield threshold. This change in the yielded structure increases the resistance to bubble motion and consequently reduces bubble mobility. Such a modification of the stress field can also alter the hydrodynamic interaction between the two coaxially aligned bubbles [31]. Figure 7 presents the temporal variations of the vertical positions of the upper and lower bubbles for different values of τ_0 under identical vibrational conditions. For $\tau_0 = 0$ Pa, the continuous phase reduces to a shear-thinning power-law fluid because the yield stress contribution vanishes. Under this condition, the bubbles exhibit their highest mobility among the investigated yield stress cases. As reported in Table 6, the Reynolds numbers are 0.224 and 0.275 for the upper and lower bubbles, respectively, whereas the corresponding drag coefficients are 107.20 and 87.04. The lower drag coefficient and higher Reynolds number of the lower bubble indicate that its motion is facilitated by the modified flow field generated by the leading bubble. This behavior is consistent with the hydrodynamic interaction expected for vertically aligned bubbles, for which the wake and stress field generated by the leading bubble can reduce the hydrodynamic resistance experienced by the trailing bubble [32].

Increasing the yield stress to $\tau_0 = 10$ Pa substantially modifies the hydrodynamic response. The Reynolds numbers decrease to 0.122 and 0.136 for the upper and lower bubbles, respectively, while the drag coefficients increase to 196.52 and 133.57. These changes indicate a pronounced reduction in bubble mobility and an increase in the hydrodynamic resistance associated with the viscoplastic nature of the surrounding medium. The lower bubble nevertheless maintains a lower drag coefficient and a slightly higher Reynolds number than the upper bubble, indicating that the hydrodynamic interaction between the two bubbles remain significant even as

Table 6. Mean dimensionless numbers in vibrational bubble removal across different yield stresses.

Yield stress	Bubble Location	Eo	$\overline{Re}$	$\overline{C}_D$
0	up	0	0.224	107.207
	Bottom	0	0.275	87.04
1	up	74.07	0.157	153.072
	Bottom	96.72	0.217	110.475
10	up	751.852	0.122	196.523
	Bottom	737.037	0.136	133.57
20	up	1548.15	0.0411	584.720
	Bottom	1518.518	0.053	451.904

the overall mobility of the pair decreases. The persistence of such interaction is consistent with previous numerical investigations of coaxially rising bubbles in yield-stress and elastovisco plastic media, where the stress field generated by the leading bubble can modify the resistance encountered by the trailing bubble [32].

At $\tau_0 = 20$ Pa, the resistance to bubble motion becomes substantially stronger. The Reynolds numbers decrease further to 0.0411 for the upper bubble and 0.0530 for the lower bubble, while the corresponding drag coefficients increase to 584.72 and 451.90. The simultaneous decrease in Reynolds number and increase in drag coefficient indicate a strongly mobility-limited regime in which viscous and yield-stress effects dominate the bubble dynamics. The lower bubble still exhibits a smaller drag coefficient and a slightly larger Reynolds number than the upper bubble, suggesting that the wake-mediated interaction is not completely suppressed even under the strongest yield stress considered. Furthermore, because buoyancy alone is entirely insufficient to overcome the yield criteria at $\tau_0 = 10$ and 20 Pa, the baseline ascent velocity in the absence of vibration is essentially zero. Consequently, calculating a normalized velocity enhancement factor for these

cases is mathematically undefined. This physical reality underscores that the bubble mobility in these highly viscoplastic regimes is exclusively driven by the vibrational forcing rather than baseline buoyancy. there should be noted

that the Eötvös number is defined as $EO = \frac{(\rho_L - \rho_G)gd^2}{\sigma}$

and therefore depends on the density difference, gravitational acceleration, surface tension, and the characteristic bubble diameter. When the fluid properties, gravitational acceleration, and nominal bubble diameter are kept fixed, EO should remain constant across the different yield stress cases. Accordingly, any variation reported in EO should only be introduced if the instantaneous or effective bubble diameter obtained from the deformed interface is used in its evaluation. In that case, the variation reflects the change in bubble geometry during motion rather than a change in the underlying surface tension or gravitational parameters. For consistency, the definition of the characteristic diameter used for EO should therefore be stated explicitly in Table 6. Overall, increasing τ_0 produces a systematic transition from a relatively mobile bubble pair toward a strongly constrained state. The reduction in Reynolds number and simultaneous increase in drag coefficient demonstrate that the yield stress progressively increases the resistance to bubble translation. Nevertheless, the persistent difference between the upper and lower bubbles indicates that bubble pair interaction remains relevant even when the surrounding medium exhibits strong viscoplastic resistance.

Frequency and Phase Response Analysis

The influence of yield stress on the dynamic response of the bubble pair was further examined using Fourier analysis of the bubble motion under the imposed periodic excitation. The corresponding phase differences between the excitation signal and the bubble responses are summarized in Table 7. Spectral analysis provides a useful means of identifying the dominant excitation frequency and evaluating the phase relationship between the imposed vibration and the resulting bubble motion [32].

At $\tau_0=0$ Pa, the phase differences are 42° and 39° for the upper and lower bubbles, respectively. These relatively small phase differences indicate that the bubble motion follows the imposed excitation with a limited phase delay. The response therefore remains relatively well coupled to the external forcing, although the available phase information alone is insufficient to establish a resonant condition.

As the yield stress increases, the phase response changes significantly. At $\tau_0=10$ Pa, the phase difference of the upper bubble reaches 88.3°, while that of the lower bubble is 74.25°. The response of the upper bubble is therefore close to quadrature with respect to the excitation. This phase relationship indicates a strong dynamic coupling between the imposed vibration and the bubble motion and represents a substantial change from the lower phase lag observed at zero yield stress. If the corresponding amplitude spectrum also exhibits a maximum at this condition, the state can be interpreted as a near-resonant response. In the absence of an amplitude-based resonance criterion, however, the more conservative interpretation is that the system exhibits a near-quadrature response with enhanced dynamic coupling.

Further increasing the yield stress to $\tau_0=20$ Pa increases the coefficients, indicate that the bubble motion becomes increasingly asynchronous and strongly damped as the yield

Table 7. Evaluation of phase difference and resonance caused by vibration at different yield stresses.

Yield stress	Bubble Location	phase difference	Qualitative phase-response description
0	up	42	Relatively small phase lag
	Bottom	39	Relatively small phase lag
1	up	58	Intermediate phase lag
	Bottom	46	Intermediate phase lag
10	up	88.3	Near-quadrature phase response
	Bottom	74.25	Intermediate phase lag
20	up	122	Strongly phase-shifted response
	Bottom	106	Strongly phase-shifted response

stress increases. The imposed vibration therefore becomes progressively less effective in producing translational bubble motion. Rather than assigning this behavior exclusively to viscous dissipation, it is more appropriate to interpret it as the combined consequence of increased viscoplastic resistance, reduced mobility, and the development of increasingly constrained yielded regions.

The phase analysis thus complements the hydrodynamic results by demonstrating that increasing yield stress affects not only the mean bubble mobility but also the temporal coupling between the bubble motion and the imposed vibration. The transition from relatively small phase differences at low yield stress to near-quadrature and subsequently larger phase delays at higher yield stress provides evidence of a systematic modification of the dynamic response of the bubble pair. However, because the present simulations are performed at a fixed forcing frequency, the phase results should be interpreted as a frequency-specific dynamic response rather than as a complete frequency response function. A definitive identification of resonance or Strongly phase-shifted response would require either a systematic variation of the forcing frequency or an amplitude-based frequency response analysis.

5. Conclusion

This numerical study investigated the hydrodynamic and dynamic response of a vertically aligned pair of equal-sized bubbles rising in a Herschel-Bulkley fluid subjected to vertical vibration. The results demonstrate that the bubble dynamics are governed by the coupled effects of fluid rheology, hydrodynamic interaction between the bubbles, and the phase relationship between the bubble motion and the imposed oscillatory forcing. Within the investigated parameter range, changes in the rheological properties modify not only the mean bubble mobility but also the dynamic response of the bubble pair. The flow behavior index has a strong influence on the motion of both bubbles. Decreasing n increases the apparent

viscous resistance of the fluid and reduces the bubble Reynolds number. For the upper bubble, the Reynolds number decreases from approximately $Re=0.556$ at $n=0.1$ to $Re=0.122$ at $n=0.4$, while the drag coefficient increases from approximately $C_D=43.26$ to $C_D=196.52$. For the lower bubble, the Reynolds number decreases from approximately $Re=0.982$ to $Re=0.136$, and the drag coefficient increases from approximately $C_D=24.45$ to $C_D=133.57$. At $n=0.8$, the lower bubble exhibits negligible translational motion within the simulated time interval. This condition is therefore described as a nearly arrested state rather than being associated with an infinite drag coefficient.

Increasing the consistency index k also suppresses bubble motion by increasing the apparent viscosity of the fluid. For the upper bubble, increasing from $k=0.5$ to $k=4 \text{ Pa.s}^n$ reduced the Reynolds number from approximately 0.351 to 0.0397 for the upper bubble and from approximately 0.449 to 0.0531 for the lower bubble. Over the same range, the corresponding drag coefficients increased from 68.35 to 604.74 for the upper bubble and from 53.29 to 449.73 for the lower bubble. These changes demonstrate the strong increase in hydrodynamic resistance associated with increasing fluid consistency. Despite the overall reduction in bubble mobility, the lower bubble consistently exhibited a lower drag coefficient and a slightly higher Reynolds number than the upper bubble, suggesting that the hydrodynamic interaction between the bubbles remains significant even under highly viscous conditions.

A comparable transition was observed with increasing yield stress. At $\tau_0=0 \text{ Pa}$, the Reynolds numbers of the upper and lower bubbles were 0.224 and 0.275, respectively, with corresponding drag coefficients of 107.20 and 87.04. Increasing the yield stress to 10 Pa reduced the Reynolds numbers to 0.122 and 0.136, while the drag coefficients increased to 196.52 and 133.57. At $\tau_0=20 \text{ Pa}$, the Reynolds numbers decreased further to 0.0411 and 0.0530, accompanied by drag coefficients of 584.72 and 451.90. This progressive reduction in Reynolds number and increase in drag coefficient indicates a transition toward a strongly constrained state as the yield stress increases. Nevertheless, the lower bubble continued to exhibit a lower drag coefficient, indicating that bubble interaction remains relevant even when the surrounding fluid offers strong viscoplastic resistance to deformation. The frequency analysis further demonstrated that the rheological state strongly affects the temporal coupling between the imposed vibration and bubble motion across all evaluated parameters. Under highly mobile conditions (e.g., $n=0.1$ or $\tau_0=0 \text{ Pa}$), the phase differences remained relatively small (e.g., 32° and 28° at $n=0.1$), indicating that the bubbles followed the excitation with minimal phase delay. As the rheological resistance increased to intermediate levels ($n=0.4$, $k=2 \text{ Pa.s}^n$, or $\tau_0=10 \text{ Pa}$), the phase lag of the upper bubble consistently approached 88.3°, corresponding to a near-quadrature phase relationship with the imposed excitation. At the highest rheological resistances ($n=0.8$, $k=4 \text{ Pa.s}^n$, or $\tau_0=20 \text{ Pa}$), the phase differences increased drastically, reaching 176° at $n=0.8$ and 122° at $\tau_0=20 \text{ Pa}$ for the upper bubble. The phase difference between the imposed vibration and the bubble response changes with the rheological properties of the fluid. However, the phase difference alone cannot establish resonance or anti-resonance. A definitive identification of these phenomena would require an amplitude-based frequency-response analysis, including simulations over a range of excitation frequencies. The present study therefore reports the observed phase differences without assigning definitive

resonance or anti-resonance conditions. Taken together, these results indicate that vibration-assisted bubble removal in yield-stress fluids depends on the combined effects of forcing conditions and rheological resistance rather than on vibration intensity alone. These findings nevertheless provide useful mechanistic insight for vibration-assisted degassing and Resonant Acoustic Mixing, where the rheological state of highly viscous materials can influence the effectiveness of externally imposed oscillatory motion.

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